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Bounding immersed curves

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Abstract

This paper addresses the immersion problem raised by H. Hopf and R. Thom in Séminaire Bourbaki 1957–58 (Exposé 157) as reported in V. Poénaru (1967/68). Let γ be a smooth closed connected curve on a surface M . The immersion problem is: when does γ extend to an immersion in M of a surface with boundary equal to γ ? We study curves on closed orientable surfaces and give a solution in the case that the genus $g \neq 1$. © 1997 Elsevier Science B.V.

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AMS classification: 57M99

1. Introduction

Progress on the problem is reported on in [1,4–7,9,11,12]. In particular there is a complete algorithmic solution when M is the plane or the sphere (as long as the immersion is not a surjection).

Let M be a closed orientable surface. We consider closed normal curves on M . (A normal curve is just a curve in general position. It has a finite number of self-intersections that are double points at which the curve crosses itself.) Now normal curves γ_1 and γ_2 on closed orientable surfaces M_1 and M_2 respectively, are said to be *geotopic* [3] if there is a homeomorphism, say $g: M_1 \rightarrow M_2$ for which $g \circ \gamma_1 = \gamma_2$. And the curves γ_1 and γ_2 are said to be *stably geotopic* [3] if one can surge a finite collection of handles from the complement of the curve in M_1 (or M_2), such that γ_1 is geotopic to γ_2 on the surgered surfaces.

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Given a normal closed curve γ on an arbitrary closed oriented surface M one can describe a CW-complex determined by γ , in the following way. The points of self-intersection of γ are the 0-cells, the open arcs are the 1-cells. Now the connected components of the complement of γ in M are not necessarily simply-connected but they have an orientation induced from the orientation of the surface, so the boundary maps are well-defined. A *filling* immersed curve [3] on a surface M is an immersed curve for which the complement of the curve in the surface is a finite collection of simply-connected components. If the curve γ is not a filling curve, then a filling curve γ^* can be obtained from γ : one simply surgers each handle that is in a connected component of $M \setminus \gamma$. Clearly γ^* is stably geotopic to γ . Let M_γ be the surface obtained from M in this way. The surface M_γ is the minimal genus closed surface on which a filling curve equivalent to γ under stable geotopy is realizable. (See [3] in which a minimal genus surface is constructed from the signed intersection sequence of a curve and a one to one correspondence is given between stable geotopy classes of normal curves on orientable compact surfaces and the isomorphism classes of abstract intersection sequences.) The 2-cells of the CW-complex determined by γ , will be the connected components of $M_\gamma \setminus \gamma^*$. The surface M_γ is the *faceted* surface of γ . We explain shortly why the immersion problem for γ can be reduced to that of the curve γ^* .

First, let γ be an immersed curve and let f_1, f_2, \dots, f_r be the simply-connected faces of the CW-complex of M_γ described above. If γ extends to an immersion $\Gamma: N \rightarrow M_\gamma$ of a surface N then γ must be the boundary of a sum $c_1 f_1 + c_2 f_2 + \dots + c_r f_r$ of the faces of M_γ , where for each i the integer c_i is equal to the cardinality of the set $\Gamma^{-1}(x)$ for any point $x \in f_i$. That is to say, γ must be nullhomologous on M_γ when considered as a 1-cycle. Indeed a normal curve γ is nullhomologous on the surface M_γ if and only if the simply-connected faces of M_γ can be numbered such that at any point of γ the number to the left is one more than the number to its right (see, for example, [2] and [8]). Evidently for a normal nullhomologous curve γ on M_γ for which

$$\gamma = \partial(c_1 f_1 + c_2 f_2 + \dots + c_r f_r),$$

the integers c_1, c_2, \dots, c_r give such a numbering. Notice also that for a closed surface such a numbering is unique up to the addition of a constant. Let γ be a nullhomologous curve and let $\{c_1, c_2, \dots, c_r\}$ be a numbering. Let $m_0 = \min\{c_1, c_2, \dots, c_r\}$ and let $m_h = \max\{c_1, c_2, \dots, c_r\}$, then

Definition. The *height* h of γ is the difference $m_h - m_0$.

Suppose that M has genus g and that the genus of the faceted surface M_γ is g_m . So M_γ is obtained by surgering $g - g_m$ handles from $M \setminus \gamma$. Now if γ bounds an immersed surface on M then γ is nullhomologous, so γ^* is nullhomologous on M_γ , where γ^* is stably geotopic to γ . Furthermore the surgery just reduces the genus of any region of M from which a handle is surgered so one has that the curve γ^* bounds an immersed surface (possibly of lower genus) on M_γ . Conversely, suppose that γ^* bounds an immersed surface. Now M is obtained from M_γ by attaching handles. If a handle is attached to regions of M_γ which are unequally numbered by the immersion, then γ is not nullho-

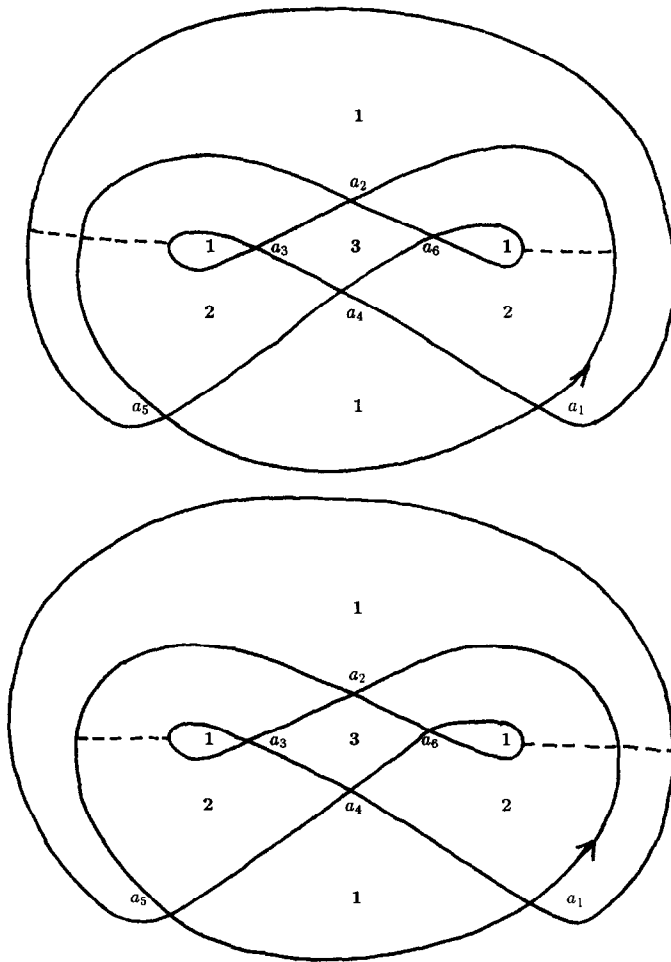


Fig. 1. Milnor's planar curve.

mologous on M and so cannot bound an immersed surface. But if M is obtained from M_γ by attaching handles to equally numbered components, then γ bounds an immersed surface (possibly of higher genus) on M . Thus we reduce the problem to:

Problem. When does a normal nullhomologous curve γ on its faceted surface M_γ extend to the immersion of a surface with boundary equal to γ ?

Two immersions Γ_1 and Γ_2 of a surface N into a closed orientable surface M are said to be different if there does not exist an orientation-preserving diffeomorphism $d: M \rightarrow M$ say, such that $\Gamma_1 = d \circ \Gamma_2$. Milnor gave an example of a planar curve γ which bounds two different immersions of the disc. This example was generalized in [9]. We show Milnor's example in Fig. 1; the dashed lines indicate how the immersed surface can be cut into embedded pieces. The reader will notice that in this example we have the same

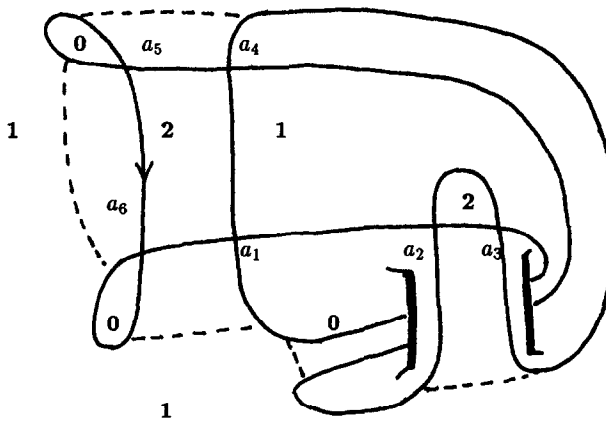


Fig. 2a. Facetted surface of genus 1; immersed surface of genus 1; nonsurjective immersion.

numbering of the faces of the surface for each immersion, where the numbering is given in bold type and the vertices are labeled a_1, a_2, \dots, a_6 respectively. In Fig. 2a we give an example in which the facetted surface is the torus. As in Milnor's example the given curve bounds two different immersions of a surface. In Fig. 2b we give a sequence of diagrams to indicate the immersed surface. Again the dashed lines represent cuts. The reader will notice that by cutting along the dashed line from the point of γ labeled c to the point labeled d in the first diagram of Fig. 2b, one creates a part boundary of a region ζ say, which is homeomorphic to a surface (with boundary) of genus one. The boundary of ζ is completed by the oriented simple arcs from d to a_5 , from a_5 to a_6 and from a_6 to c . The dot-dash lines represent a regular homotopy of the curve. Cut out ζ then move the simple arc joining c to d by a regular homotopy, to the dot-dash line. This cuts off a simply-connected region. In the third diagram of the figure the reader can see how to cut off another two simply-connected regions, reducing the immersed surface to that indicated in the final diagram of the figure. Unlike Milnor's example here we have a distinct numbering given by each immersion. Now G.K. Francis [4] gave an example of a curve which bounds two different surfaces immersed in the sphere. We find that for each of these surfaces the immersion gives a different numbering to the faces of the facetted surface. These examples motivate the following definition.

Let $I(\gamma)$ be the set consisting of pairs (Γ_N, N) where Γ_N is an immersion of a compact surface N into the facetted surface M_γ of γ such that $\gamma = \Gamma_N|_{\partial N}$. For $(\Gamma_N, N) \in I(\gamma)$, let $\{c_1, c_2, \dots, c_r\}$ be the set of nonnegative integers assigned by the immersion Γ_N , to the faces of M_γ . Let μ_{Γ_N} be the maximum of the set $\{c_1, \dots, c_r\}$ for an immersion Γ_N of N .

Definition. The *immersion number* μ_γ of γ is $\max_{I(\gamma)} \{\mu_{\Gamma_N}\}$.

Notice that $\mu_\gamma \geq h$, where h is the height of γ . Now in Figs. 3–5 we give 3 examples in which $\mu_\gamma > h$. In each case one has $\mu_{\Gamma_N} = 3$ and $h = 2$. The curve in Fig. 3 has facetted surface M_γ of genus zero. That in Fig. 4 has genus two and in Fig. 5 the facetted

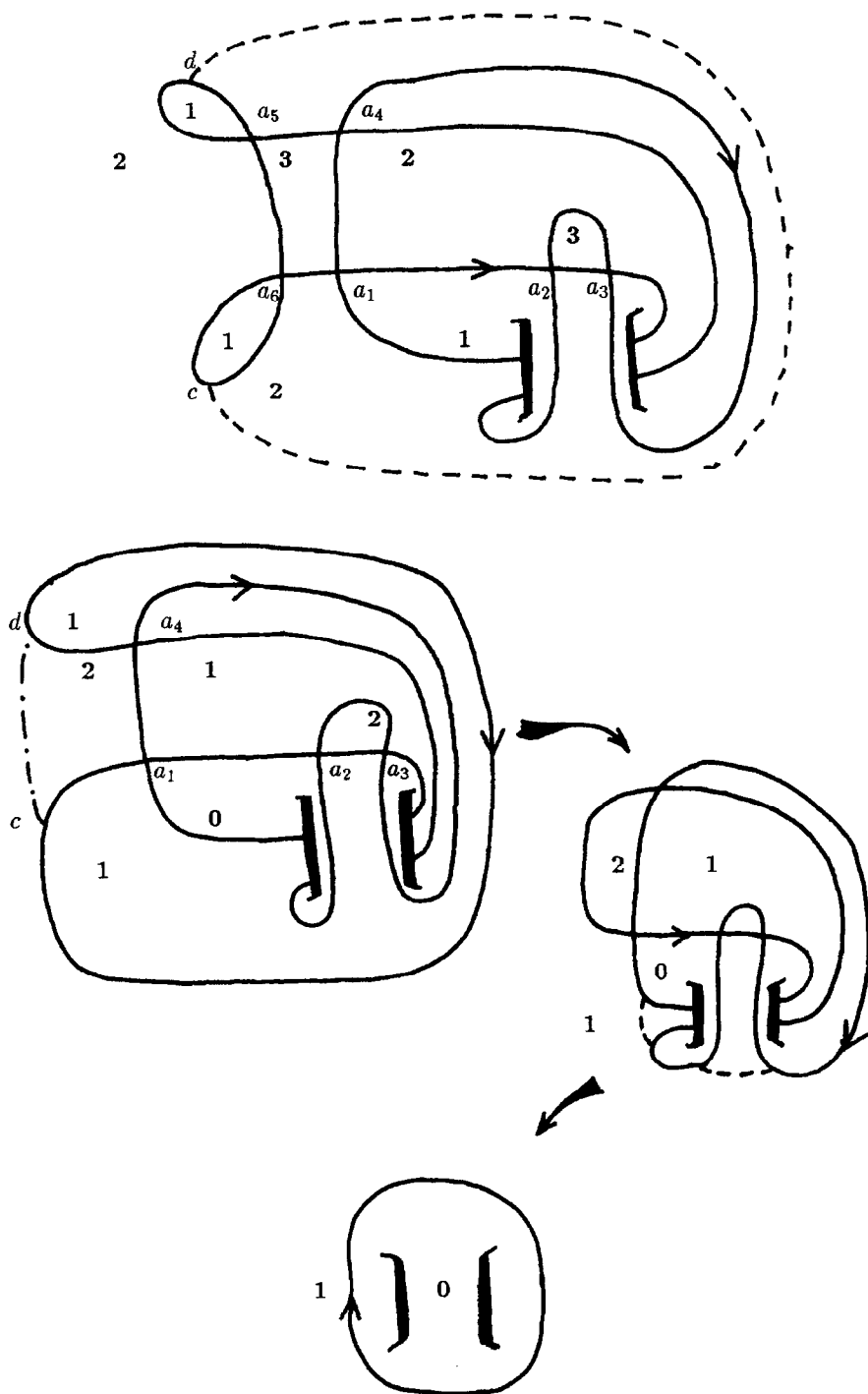


Fig. 2b. Facetted surface of genus 1; immersed surface of genus 1; surjective immersion.

surface of the given curve has genus one. We give a sequence of diagrams which show that with $\mu_{\Gamma_N} = 3$ the curve bounds an immersed surface. We leave it to the reader to check that the curve does not bound an immersed surface for $\mu_{\Gamma_N} = 2$.

The following theorems give an upper bound on the immersion number.

Theorem 1. *Let γ be a curve with k crossings and height h that bounds an immersed surface. If the faceted surface M_γ of γ is the sphere, then the immersion number μ_γ of γ is bounded above by*

$$\frac{(h-1)k}{2} + 1.$$

Theorem 2. *Let γ be a curve with k crossings and height h on the surface M_γ . And suppose that γ bounds an immersed surface N . If M_γ has genus greater than one, then the immersion number μ_γ is less than or equal to*

$$\left\lfloor \frac{(h-1)(k-h) - \chi(N)}{|\chi(M_\gamma)|} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of x and χ denotes the Euler characteristic.

The following finite algorithm is a solution to the problem. Let γ be a normal nullhomologous curve with k crossings and height h . For a nonnegative numbering $\{c_1, \dots, c_r\}$ of the faces of M_γ , for each i take c_i copies of the face f_i and consider all possible ways of assembling these faces to form a surface N . For a given numbering this is obviously a finite algorithm. One then repeats the process for each nonnegative numbering for which $\max\{c_1, \dots, c_r\}$ is no greater than the bound given by Theorem 1, in the case that $M_\gamma = S^2$; or by Theorem 2, in the case that M_γ has genus ≥ 2 . In the case of curves which have a faceted surface of genus one a finite algorithm is not possible.

2. Proof of the theorems

Let γ be a normal, nullhomologous curve with faceted surface M_γ . Suppose that γ has k crossings and is of height $h \geq 1$. Let $\Gamma: N \rightarrow M_\gamma$ be an immersion of a connected compact surface N into M_γ such that the curve $\Gamma|_{\partial N}$ is equal to γ , where ∂N is the boundary of N . Let W be the CW-complex of M_γ described in the introduction. And let $\chi(M_\gamma)$ be the Euler characteristic of M_γ given by the sum $V - E + F$, where V is the number of vertices, E is the number of edges and F is the number of faces of W . We have that $E = 2V = 2k$. So $F = \chi(M_\gamma) + k$. Let the set $\{f_1, \dots, f_{\chi(M_\gamma)+k}\}$ be the faces of W . Then, for each integer i such that $1 \leq i \leq \chi(M_\gamma) + k$, we assign the nonnegative integer $\#\Gamma^{-1}(x)$ to f_i , where $x \in f_i$.

Let $\{m_0, \dots, m_h\}$ be the ordered set of nonnegative integers so assigned to the faces of W . We say that $\{m_0, \dots, m_h\}$ is the numbering determined by Γ . The height h equals $m_h - m_0$.

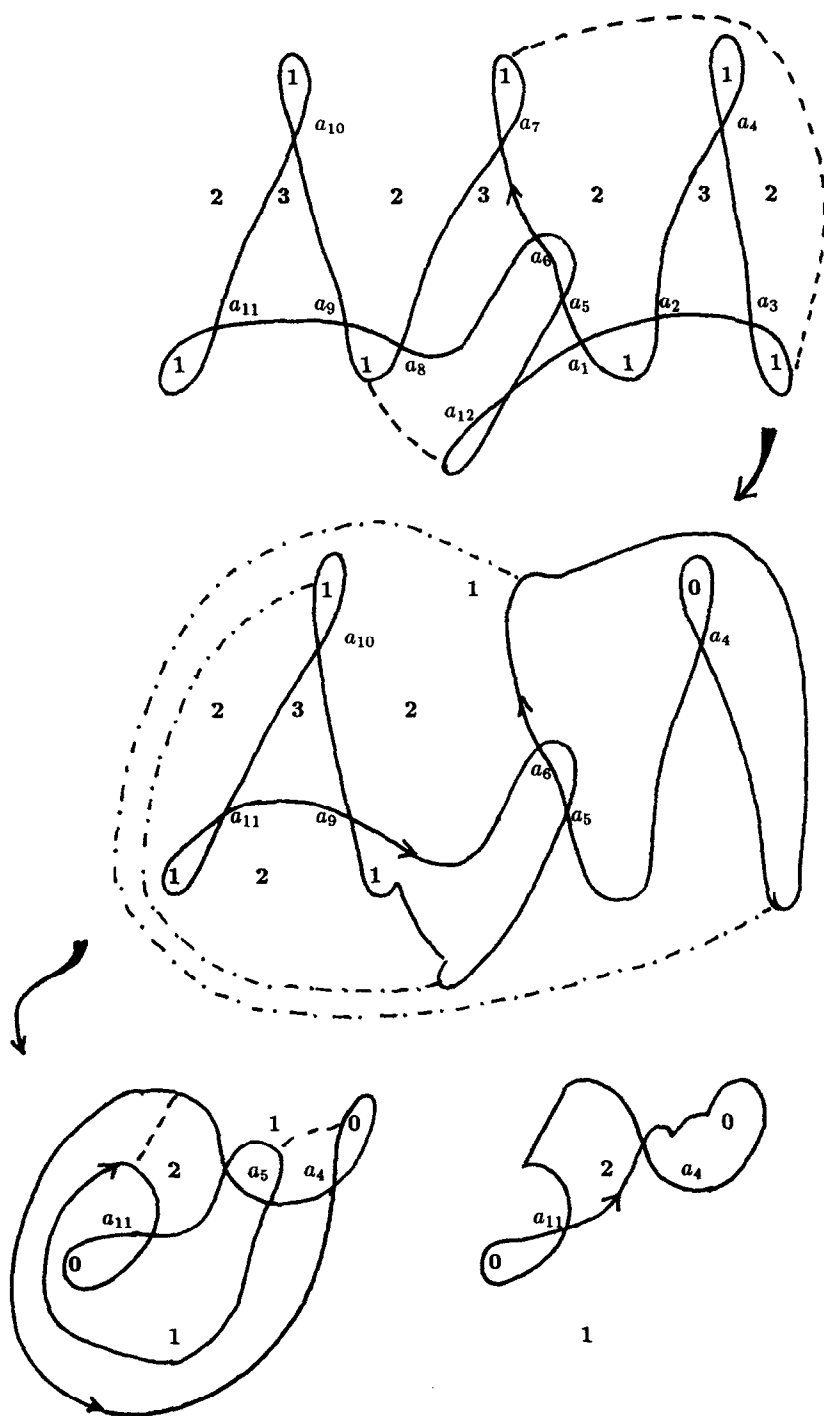


Fig. 3. Facetted surface of genus 0; immersed surface of genus 1; surjective immersion.

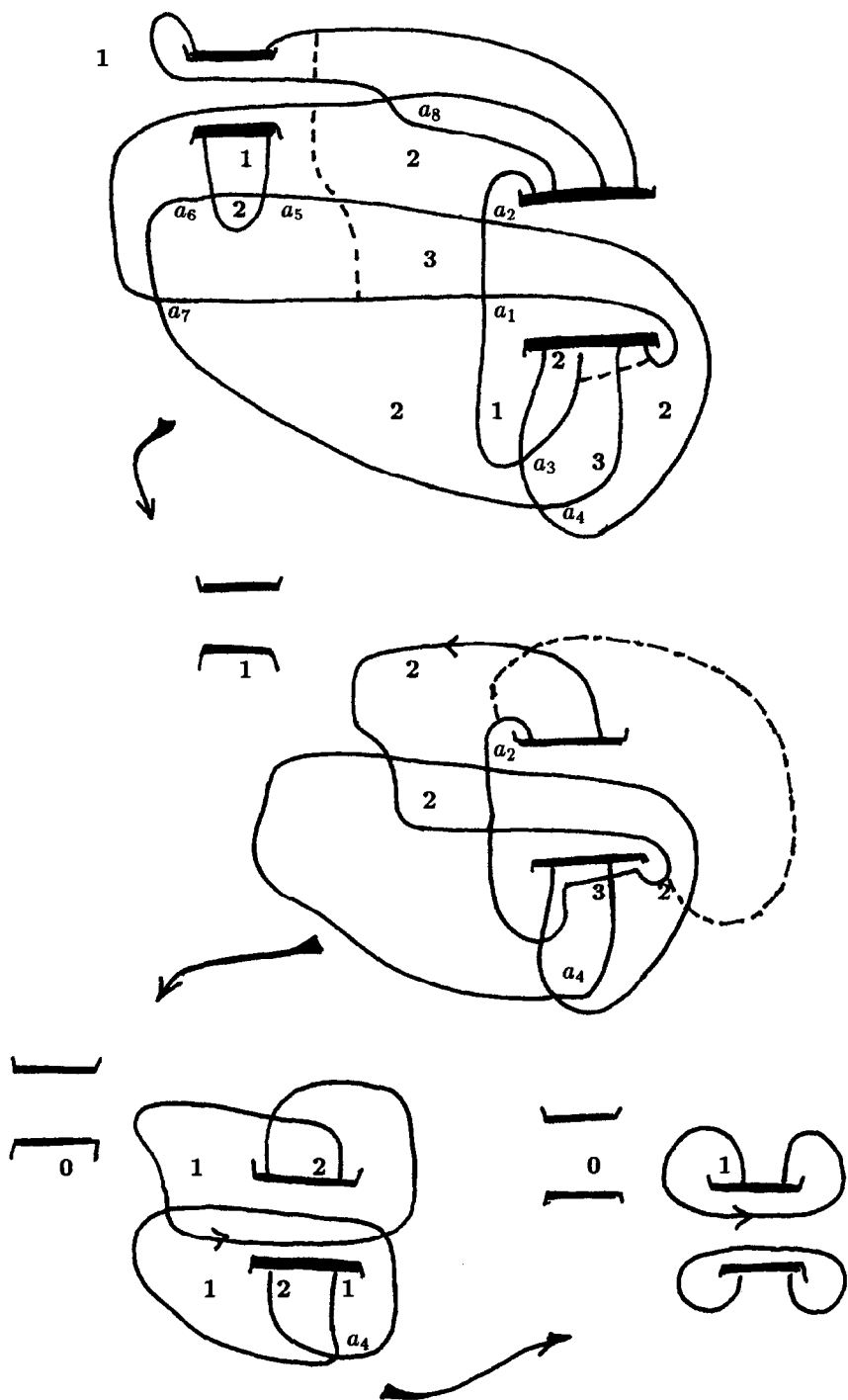


Fig. 4. Facetted surface of genus 2; immersed surface of genus 2; surjective immersion.

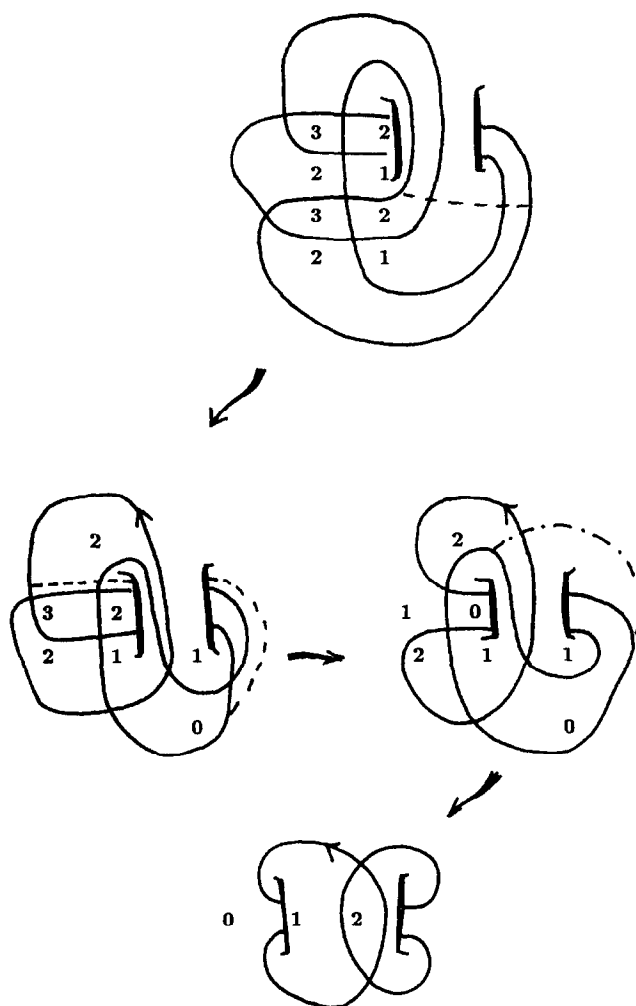


Fig. 5. Facetted surface of genus 1; immersed surface the disc; surjective immersion.

The following three lemmas are central to our proofs of the theorems. For each non-negative integer j , let S_j be the closure of the union of faces which are numbered with an integer greater than or equal to j . Then:

Lemma 3.

$$\chi(N) = \sum_{j=1}^{m_h} \chi(S_j).$$

Remark. That the winding number of γ is $\chi(N)$, taken modulo $|\chi(M_\gamma)|$ is a result attributed to A. Haefliger in [10]. And by [8] the winding number of γ is $\sum_{j>0} \chi(S_j)$,

again taken modulo $|\chi(M_\gamma)|$. So the content of Lemma 3 is that one has equality of the integers $\chi(N)$ and $\sum_{j>0} \chi(S_j)$, rather than equality modulo $|\chi(M_\gamma)|$.

Lemma 4. *Suppose that the faceted surface M_γ is the sphere. Then*

- (1) $1 \leq \chi(S_{m_h})$,
- (2) $h - k \leq \chi(S_{m_i})$, and
- (3) for each $2 \leq i \leq h - 1$ one has $1 - k \leq \chi(S_{m_i})$.

Remark. If a curve γ on the sphere has height one, then the number of crossings of γ is zero. So the curve γ is a simple Jordan curve that bounds a disc embedded in the sphere.

Lemma 5. *In general, one has*

- (1) $1 \leq \chi(S_{m_h})$,
- (2) $\chi(S_{m_i}) < \chi(M_\gamma)$, and
- (3) for each $0 \leq i \leq h$ one has $\chi(S_{m_i}) \leq \chi(M_\gamma) + k - h$.

Proof of Lemma 3. Each surface S_j is a subcomplex of the CW structure, W , on M_γ . The 0-cells are those crossings of γ that are in the boundary of any face of W numbered greater than or equal to j , the 1-cells are the arcs of γ that are in the boundary of any face numbered greater than or equal to j and the 2-cells are the faces numbered greater than or equal to j . One has the containment

$$S_{m_h} \subset S_{m_{h-1}} \subset \cdots \subset S_{m_0} = \cdots = S_0 = W,$$

where $\{m_0, \dots, m_h\}$ is the ordered set of nonnegative integers assigned to the faces of W and determined by the immersion Γ .

For each $1 \leq j \leq m_h$, the surface $\overline{S_j \setminus S_{j+1}}$ is the closure of the complement of S_{j+1} in S_j . Thus the surface $\overline{S_j \setminus S_{j+1}}$ is the closure of the union of faces numbered exactly j . And since no face is numbered $m_h + 1$ we have that the closure of $S_{m_h} \setminus S_{m_h+1}$ is equal to S_{m_h} . Now for each integer i , the intersection $\overline{S_i \setminus S_{i+1}} \cap S_{i+1}$ is a union of simple closed curves. So the Euler characteristic $\chi(\overline{S_i \setminus S_{i+1}} \cap S_{i+1}) = 0$. Hence, for each $1 \leq j \leq m_h$,

$$\begin{aligned} \chi(S_j) &= \chi\left(\bigcup_{i=j}^{m_h} (\overline{S_i \setminus S_{i+1}})\right) \\ &= \chi(\overline{S_j \setminus S_{j+1}}) + \chi(\overline{S_{j+1} \setminus S_{j+2}}) + \cdots + \chi(\overline{S_{m_h-1} \setminus S_{m_h}}) + \chi(S_{m_h}). \end{aligned}$$

And so

$$\sum_{j=1}^{m_h} \chi(S_j) = \sum_{j=1}^{m_h} \left(\sum_{i=j}^{m_h} \chi(\overline{S_i \setminus S_{i+1}}) \right) = \sum_{j=1}^{m_h} j \chi(\overline{S_j \setminus S_{j+1}}).$$

Let p_j be the number of faces, let l_j be the number of edges and let n_j be the number of vertices in $\overline{S_j \setminus S_{j+1}}$. So one has that

$$\sum_{j=1}^{m_h} \chi(S_j) = \sum_{j=1}^{m_h} j(p_j - l_j + n_j). \quad (*)$$

Consider a face ζ of W numbered i . The restriction to ζ of the immersion $\Gamma: N \rightarrow M_\gamma$ is an i -fold covering. Hence, since ζ is simply connected, the set $\Gamma^{-1}(\zeta)$ has i connected components, each homeomorphic to ζ . Let Y be the natural CW-decomposition of N whose 2-cells are the connected components of the sets $\Gamma^{-1}(\zeta)$ for the various faces ζ of W . Each oriented 1-cell in the boundary of a face of W numbered i is covered by i disjoint oriented 1-cells and similarly each 0-cell in the boundary of a face of W numbered i is covered by i 0-cells.

Let v , e and f be the number of vertices, edges and faces respectively, of the complex Y . One has that

$$f = \sum_{j=1}^{m_h} j p_j. \quad (1)$$

Let $\{f_1^j, f_2^j, \dots, f_{jp_j}^j\} = \Gamma^{-1}(\overline{S_j \setminus S_{j+1}})$. Now l_j is the number of edges in $\overline{S_j \setminus S_{j+1}}$ so jl_j gives the number of edges in the disjoint union $\coprod_{i=1}^{jp_j} f_i^j$. Thus $\sum_{j=1}^{m_h} jl_j$ is the number of edges in the disjoint union $\coprod_{j=1}^m (\coprod_{i=1}^{jp_j} f_i^j)$. If a 2-cell ζ is to the left (respectively right) of an oriented 1-cell ε , we say that ε has positive (respectively negative) orientation in the boundary of ζ . Let l_j^+ (respectively l_j^-) be the number of edges which have positive (respectively negative) orientation in the boundary of $\overline{S_j \setminus S_{j+1}}$. So

$$l_j = l_j^+ + l_j^-. \quad (2)$$

Now jl_j^+ (respectively jl_j^-) is the number of edges which have positive (respectively negative) orientation in the boundary of the faces of $\coprod_{i=1}^{jp_j} f_i^j$. Let $l^{\partial N}$ be the number of edges in the boundary ∂N . Recall that Y is the CW structure on the immersed surface N . Since the edges in ∂N are positively oriented and each edge in the interior of Y has negative orientation in the boundary of the 2-cell to its right and positive orientation in the boundary of the 2-cell to its left, one has that

$$l^{\partial N} = \sum_{j=1}^{m_h} jl_j^+ - \sum_{j=1}^{m_h} jl_j^-.$$

And the number e of edges in Y is precisely the sum of the number of edges in ∂N and the number of edges with negative orientation. That is

$$e = l^{\partial N} + \sum_{j=1}^{m_h} jl_j^-. \quad (3)$$

Now n_j is the number of vertices in $\overline{S_j \setminus S_{j+1}}$. And in W , each vertex is in the boundary of three faces to which different integers have been assigned by the immersion Γ . Let

n_j^s be the number of vertices in W for which j is the smallest integer assigned by the immersion to the incident faces. Let n_j^x (respectively n_j^l) be the number of vertices in W for which j is the middle (respectively largest) integer assigned to the incident faces. Clearly,

$$n_j = n_j^s + n_j^x + n_j^l \quad (4)$$

for each j . Also one has that $\sum_{j=1}^{m_h} j n_j$ is the number of vertices in the disjoint union $\coprod_{j=1}^{m_h} (\coprod_{i=1}^j f_i^j)$.

But each vertex in ∂N is in the boundary of two faces, one for which the integer assigned by the immersion is the largest of the three assigned to the faces that are incident at the image of the vertex in W and the other for which the integer assigned is the middle integer. And the vertices in the interior of Y are in the boundary of three faces. So the $\sum_{j=1}^{m_h} j n_j$ counts twice each vertex in ∂N and also counts the vertices in the interior of Y on three occasions; as vertices in the boundary of faces for which j is the smallest, middle and largest integer assigned by the immersion. Now ∂N is a closed curve that is a simple 1-cycle, so there are $l^{\partial N}$ vertices in ∂N . And we want to count the vertices in the interior of Y once only. Since there is exactly one face incident at any interior vertex to which a smallest integer is assigned, $\sum_{j=1}^{m_h} j n_j^s$ is the number of vertices in the interior of Y . So one has

$$v = l^{\partial N} + \sum_{j=1}^{m_h} j n_j^s. \quad (5)$$

And from Eqs. (1), (3) and (5) one has that

$$\begin{aligned} \chi(N) &= l^{\partial N} + \sum_{j=1}^{m_h} j n_j^s - l^{\partial N} - \sum_{j=1}^{m_h} j l_j^- + \sum_{j=1}^{m_h} j p_j \\ &= \sum_{j=1}^{m_h} j (n_j - l_j + p_j) - \sum_{j=1}^{m_h} j n_j^x - \sum_{j=1}^{m_h} j n_j^l + \sum_{j=1}^{m_h} j l_j^+ \quad (\text{by (2) and (4)}) \\ &= \sum_{j=1}^{m_h} \chi(S_j) - \left(\sum_{j=1}^{m_h} j n_j^x + \sum_{j=1}^{m_h} j n_j^l - \sum_{j=1}^{m_h} j l_j^+ \right) \quad (\text{by (*)}). \end{aligned}$$

Thus it remains to show that

$$\sum_{j=1}^{m_h} j n_j^x + \sum_{j=1}^{m_h} j n_j^l - \sum_{j=1}^{m_h} j l_j^+ = 0. \quad (6)$$

The left-hand side of (6) can be evaluated by considering the contributions vertex by vertex. Consider a vertex q of any face of W . Clearly it suffices to show that

$$\sum_{j=1}^{m_h} j \hat{n}_j^x + \sum_{j=1}^{m_h} j \hat{n}_j^l - \sum_{j=1}^{m_h} j \hat{l}_j^+ = 0,$$

where \hat{n}_j^x (respectively \hat{n}_j^l) is the number of times q is in the boundary of a face numbered j for which j is the middle (respectively largest) integer assigned to the faces of W by

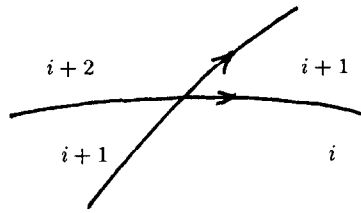


Fig. 6. A local vertex.

Γ and \hat{l}_j^+ is half the number of ends of edges incident at q that are positively oriented in the boundary of a face numbered j . Then summing over all of the vertices gives the required result.

Let i be the smallest number assigned to the faces incident at q . See Fig. 6. Now the vertex q occurs once in the boundary of the face numbered $i+1$ and once in the boundary of the face numbered $i+2$. Also there are two ends of positively oriented edges in the boundary of the face numbered $i+1$ and two ends in the boundary of the face numbered $i+2$. This gives

$$\begin{aligned} \sum_{j=1}^{m_h} j \hat{n}_j^x + \sum_{j=1}^{m_h} j \hat{n}_j^l - \sum_{j=1}^{m_h} j \hat{l}_j^+ \\ = i \hat{n}_i^x + (i+1) \hat{n}_{i+1}^x + (i+2) \hat{n}_{i+2}^x + i \hat{n}_i^l + (i+1) \hat{n}_{i+1}^l + (i+2) \hat{n}_{i+2}^l \\ - i \hat{l}_i^+ - (i+1) \hat{l}_{i+1}^+ - (i+2) \hat{l}_{i+2}^+ \\ = 0 + (i+1) + 0 + 0 + 0 + (i+2) - 0 - (i+1) - (i+2) \\ = 0, \quad \text{as required.} \end{aligned}$$

Hence $\chi(N) = \sum_{j=1}^{m_h} \chi(S_j)$, completing the proof of Lemma 3. \square

Proof of Lemma 4. Let the faceted surface M_γ of γ be the sphere \mathbf{S}^2 . Since m_h is the maximum of the numbering, the 2-cells in S_{m_h} have disjoint closures. So the Euler characteristic $\chi(S_{m_h})$ of the surface S_{m_h} equals $\#\{2\text{-cells numbered } m_h\}$. In particular, $\chi(S_{m_h}) \geq 1$.

Now by the Euler formula, the number of 2-cells of M_γ is $2+k$ and for all $1 \leq i \leq h$, the set S_{m_i} contains at least one 2-cell, so the number of 2-cells numbered m_0 is bounded above by $2+k-h$. Since m_0 is the minimum of the numbering of 2-cells one has

$$\begin{aligned} 2 = \chi(\mathbf{S}^2) &= \chi(S_{m_1}) + \#\{2\text{-cells numbered } m_0\} \\ &\leq \chi(S_{m_1}) + 2 + k - h. \end{aligned}$$

Thus $\chi(S_{m_1}) \geq h-k$.

It remains to prove (3) of the lemma, in which we establish a lower bound on $\chi(S_{m_i})$ for each $2 \leq i \leq h-1$. Now fix i and suppose that S_{m_i} has c_i connected components and that $\chi(S_{m_i}) < c_i - k$. The surface S_{m_i} in \mathbf{S}^2 may be viewed as c_i discs with holes. Each disc has piecewise-smooth boundary and by the assumption on the value of the Euler characteristic, the total number of holes in S_{m_i} must be greater than k . But each

hole in S_{m_i} is the union of faces which were assigned an integer smaller than m_i . And in the boundary of each hole there is at least one vertex. So if S_{m_i} has more than k holes, then the curve γ has more than k crossings. This contradicts the assumption on γ . Hence for each $2 \leq i \leq h-1$, one has $\chi(S_{m_i}) \geq c_i - k \geq 1 - k$ as required. \square

Proof of Lemma 5. Since the height of γ is h and there is at least one 2-cell numbered by each of the integers m_0, m_1, \dots, m_h , there are at most $\chi(M_\gamma) + k - h$ 2-cells numbered m_j , for any j where $0 \leq j \leq h$. It follows that for each $0 \leq j \leq h$, one has $\chi(S_{m_j}) \leq \chi(M_\gamma) + k - h$. And the Euler characteristic $\chi(S_{m_h})$ of S_{m_h} equals $\#\{2\text{-cells numbered } m_h\}$. Thus $1 \leq \chi(S_{m_h})$. The Euler characteristic $\chi(S_{m_1})$ of the surface S_{m_1} equals $\chi(M_\gamma) - \#\{2\text{-cells numbered } m_0\}$ and so $\chi(S_{m_1}) < \chi(M_\gamma)$, completing the proof of Lemma 5. \square

We return now to the proof of the theorems.

Proof of Theorems 1 and 2. Let N be the compact connected surface immersed in the faceted surface M_γ by an immersion Γ , such that the restriction of the immersion $\Gamma|_{\partial N}$ to the boundary ∂N of N equals γ . Let g_N be the genus of N . And let W (respectively Y) be the CW-complex of M_γ (respectively N). Recall that $\{m_0, \dots, m_h\}$ denotes the ordered set of integers assigned to the 2-cells of W , where h is the height of γ . Let μ_γ be the immersion number of γ . We have that

$$1 - 2g_N = \chi(N) = \sum_{j=1}^{m_h} \chi(S_j) \quad (\text{by Lemma 3}).$$

If $M_\gamma = \mathbf{S}^2$, then

$$\begin{aligned} \sum_{j=1}^{m_h} \chi(S_j) &= m_0 \chi(\mathbf{S}^2) + \chi(S_{m_1}) + \sum_{i=2}^{h-1} \chi(S_{m_i}) + \chi(S_{m_h}) \\ &\geq 2m_0 + h - k + (h-2)(1-k) + 1 \quad (\text{by Lemma 4}) \\ &= 2m_0 + 2h + k - hk - 1 \\ &= 2m_h + k - hk - 1. \end{aligned}$$

That is

$$2m_h \leq hk + 2 - 2g_N - k.$$

Now $g_N \geq 0$, so

$$m_h \leq \frac{(h-1)k}{2} + 1.$$

Since the immersion number μ_γ is $\max_{I(\gamma)} \{m_h\}$, it is bounded above by the same bound. This completes the proof of Theorem 1. \square

Remark. The curve shown in Fig. 7 is an example in which this upper bound is attained. One has that $h = 2$, $k = 4$ and so $m_h \leq (2-1)(4)/2 + 1 = 3$ by Theorem 1.

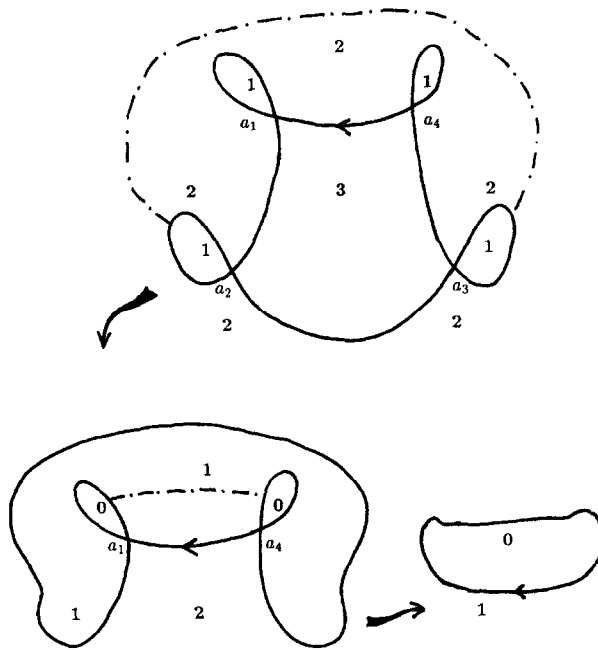


Fig. 7. An example in which the upper bound of Theorem 1 is attained.

If the genus of M_γ is greater than one, then

$$\begin{aligned} \sum_{j=1}^{m_h} \chi(S_j) &= m_0 \chi(M_\gamma) + \chi(S_{m_1}) + \sum_{i=2}^h \chi(S_{m_i}) \\ &< (m_0 + 1) \chi(M_\gamma) + (h - 1) (\chi(M_\gamma) + k - h) \quad (\text{by Lemma 5}). \end{aligned}$$

So

$$m_h \chi(M_\gamma) > 1 - 2g_N - (h - 1)(k - h).$$

Hence

$$m_h \leq \left\lfloor \frac{(h - 1)(k - h) - (1 - 2g_N)}{|\chi(M_\gamma)|} \right\rfloor,$$

where $\lfloor x \rfloor$ denotes the integer part of x . This completes the proof of Theorem 2. \square

Final remark. We have found a family Γ_g of curves γ_g which have $4g$ crossings, height two and faceted surface M_{γ_g} of genus $g \geq 2$, where for each g , the curve γ_g bounds a disc N immersed in M_{γ_g} . Since $\chi(N) = 1$, $h = 2$ and $k = 4g$ by Theorem 2 one has that

$$m_h \leq \left\lfloor \frac{(h - 1)(k - h) - (1 - 2g_N)}{|\chi(M_\gamma)|} \right\rfloor = \left\lfloor \frac{4g - 3}{2g - 2} \right\rfloor = 2 + \left\lfloor \frac{1}{2g - 2} \right\rfloor = 2$$

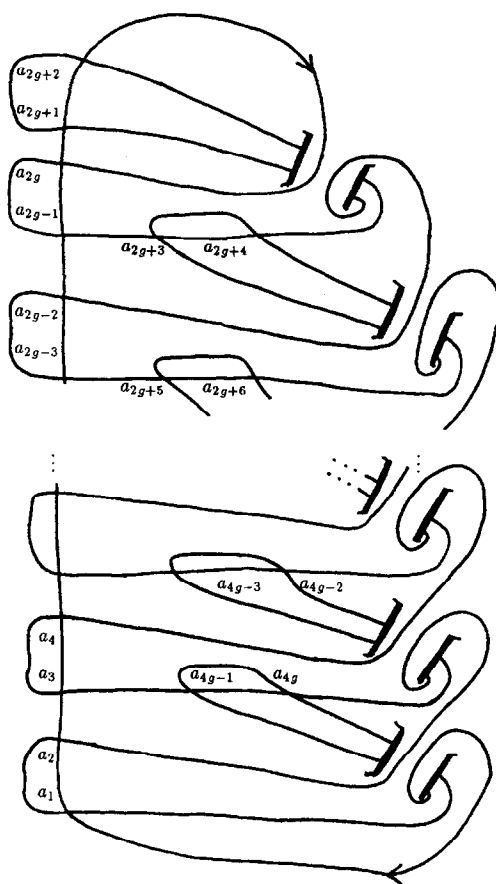


Fig. 8. An example in which the upper bound of Theorem 2 is attained.

since $g \geq 2$. Thus Γ_g is a family of curves for which the upper bound of Theorem 2 is attained. We illustrate this in Fig. 8.

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